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AN INVERSE COEFFICIENT IDENTIFICATION PROBLEM IN A DYNAMIC PLATE MODEL

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Abstract - We consider the problem of identifying the spatially varying flexural rigidity of a plate using an observed deflection solution of the direct problem of a dynamic linear Kirchhoff plate model. For this inverse problem, several uniqueness and continuous dependence results are developed.

1. INTRODUCTION

Many rocket propulsion laboratory structures are designed to serve as a test bed for the implementation and evaluation of control algorithms for the large angle slewing of spacecraft with flexible appendages. Furthermore, these structures are specifically designed to exhibit structural modes and damping characteristics representative of realistic large flexible space structures, [1]. The difficulties involved in the design of practical and efficient control laws for large spacecraft (e.g. the inherent infinite dimensionality of the system, a large number of closely spaced modal frequencies, high flexibility, a fuel-limited, hostile, highly variable environment, etc.) have stimulated research into the development of system identification and parameter estimation procedures which will yield high fidelity models. A particular area of interest involves schemes for the estimation of material parameters describing, for example, mass, inertia, and flexural rigidity or damping properties in distributed models for the vibration of viscoelastic systems-specifically, mechanical beams, plates and the like. The determination of the elastic properties of a deformable (or bending) material is one of the central problems in computational material diagnostics, [9].

The partial differential equation

$$m\frac{\partial^2 u}{\partial t^2} + \nabla^2(a\nabla^2 u) = f \tag{1}$$

has been considered at the steady-state in [12, 24] as a dynamic linear Kirchhoff plate model which takes into account moments parallel to the $x-$ and $y-$axes with no twisting. If a is a spacewise, piecewise constant function it represents the transient Kirchhoff equation for composite, isotropic, homogeneous plates in linear elasticity, [14]. The quantity u represents the deflection of the elastic plate, f is the load to which the plate is subjected, and a and m represent the flexural rigidity and the mass density per unit area of the plate, respectively. The flexural rigidity depends on the material properties of the plate, namely, Young's modulus and Poisson ratio, as well as the thickness of the plate, [22].

In practice it is sometime impossible to obtain a precise knowledge of the physical parameters a and m of the elastic system by experimental methods, whereas it is usually easier to take measurements of the deflection u and the load f. From such considerations arise the well known inverse problem of 'parameter identification', i.e. evaluating the physical parameter values by function solution measurements. Such a problem is ill-posed, since a solution, whenever it exists, need not be unique and generally does not depend continuously on the data.

One-dimensional coefficient identification problems in beam systems for constant coefficients m and a can be found in [1,25], whilst spacewise dependencies of a when $m = 1$ can be found in [13]. More general variations of a and/or m with respect to x, t and/or u have been considered in [16]. Further, two-dimensional anisotropic plates with unknown constant material characteristics have been considered in [6].

A common identification strategy is the "indirect" approach in which one minimizes, via an iterative least-squares process, the gap between a computed forward solution $u_{a,m}$ and the observed values. Nevertheless, the indirect approach is very powerful, especially for its simple computational implementation which can also allow for the practical case of pointwise measurements to be taken into account, as suggested in Section 5. However, when it comes to establishing theoretical uniqueness and stability estimates in the continuous case, as proposed in this study for the steady-state, an alternative "direct" approach involving an approximate solution of the second-order partial differential equation which involves the function a, namely,

$$\nabla^2(a\nabla^2 u) = a\nabla^4 u + 2\nabla a \cdot \nabla(\nabla^2 u) + (\nabla^2 a)(\nabla^2 u) = f \quad \text{in } \Omega \subset R^d \tag{2}$$

seems more feasible. Essentially the same equation for a also arises in the transient situation when one has a time history of u and f measurements, or if m is known *a priori*, which is sometimes assumed. A practical limitation to the direct approach for identifying a is that the coefficients in eqn.(2) involve derivatives of the measured quantity u. However, when it is feasible, then it is potentially simpler and less costly than using the indirect approach.

One-dimensional beam-type system coefficient identification problems for eqn.(2) can be found in [15]. Here we consider a higher-dimensional situation, which is based on a new extension of the work of Richter [20] for diffusive-like systems, using a systematic analysis of the inverse problem (2) in which the flexural rigidity coefficient a is to be identified on the basis of an observed pair f, u. A more realistic mathematical model based on the Love-Kirchhoff plate theory has been considered in [10] where the material properties forming the elastic plate have been identified from the complete knowledge of the Dirichlet to Neumann map.

In section 2 we establish the continuous dependence of the forward (direct) problem in which a and f are known and u, subject to appropriate boundary conditions, needs to be found. The difficulties arising with the inverse problem in which u and f are known and a has to be identified are shown in section 3 for the one-dimensional beam situation. The research core of the paper then establishes the uniqueness of a and its continuous dependence on the input data f, the boundary conditions, $\nabla^2 u$, $\nabla^3 u$ and $\nabla^4 u$. In practice this would require differentiating twice the bending moment or four times the deflection, both noisy functions, hence resembling the problem considered in [3] to recovering engineering loads from strain gauge data. A particular case concerned with the retrieval of a harmonic flexural rigidity is investigated in section 4. Finally, section 5 presents the conclusions of this research and an integral formulation which can be used for obtaining the numerical solution.

2. WELL-POSEDNESS OF THE FORWARD PROBLEM

Consider first the direct problem given by the fourth-order partial differential equation

$$L(a;u) := \nabla^2(a(x)\nabla^2 u(x)) = f(x), \quad x \in \Omega \tag{3}$$

where a and u are defined in a connected, bounded domain $\Omega \subset R^d$. We denote by n the outward normal to the boundary $\partial\Omega$ and we find it convenient to use the function spaces $L^2(\Omega)$ and $\{L^\infty(\Omega)\}^n$, whose norms are given by

$$\|f\|_{L^2(\Omega)} = \left(\int_\Omega f^2(x)dx\right)^{1/2}, \quad \|\underline{f}\|_{\{L^\infty(\Omega)\}^n} = \sum_{i=1}^{n} sup_{x\in\Omega} \mid f_i(x) \mid . \tag{4}$$

Henceforth we abbreviate these norms by $\|.\|_2$ and $\|.\|_\infty$, respectively.

We adopt the view that a good approximation to the flexural rigidity coefficient a is one which yields a good deflection solution u to the forward problem. More specifically, let u and v be two solutions of (3) which correspond to different coefficients a and b. Viewing problem (3) in its usual weak form variational setting, i.e. $u,\ v \in H^2(\Omega)$, $a,\ b \in L^\infty(\Omega)$, we ask how close must a and b be in order to guarantee that $\|\nabla^2(u-v)\|_2$ is small. The following lemma addresses this question.

Lemma 2.1. *If $L(a;u) = L(b;v)$ in Ω and*

$$u = v \quad \text{and} \quad \left(\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} \quad \text{or} \quad a\nabla^2 u = b\nabla^2 v\right), \quad \text{on } \partial\Omega \tag{5}$$

or,

$$\frac{\partial}{\partial n}\left(a\nabla^2 u\right) = \frac{\partial}{\partial n}\left(b\nabla^2 v\right) \quad \text{and} \quad \left(\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} \quad \text{or} \quad a\nabla^2 u = b\nabla^2 v\right), \quad \text{on } \partial\Omega \tag{6}$$

then, if we assume $b > 0$, we have

$$\|\nabla^2(u-v)\|_2 \leq \frac{\|a-b\|_\infty}{inf_\Omega b}\|\nabla^2 u\|_2. \tag{7}$$

Proof. First we rewrite $L(a;u) = L(b;v)$ as $L(a-b;u) = L(b;v-u)$. Multiplying this equation by $(v-u)$, integrating using Green's formulae and imposing the boundary conditions (5) or (6), we obtain

$$\int_\Omega (a-b)\nabla^2 u \nabla^2 (v-u) d\Omega = \int_\Omega b \mid \nabla^2 (v-u) \mid^2 d\Omega \tag{8}$$

Applying the Cauchy-Schwarz inequality yields

$$\|a-b\|_\infty \|\nabla^2 u\|_2 \geq \|\nabla^2(u-v)\|_2 \; inf_\Omega b \tag{9}$$

as required, see eqn.(7).

Remark 2.1. The imposition of the boundary conditions (5) and (6) corresponds to physical situations which include fixed, supported and free plates.

Based on Lemma 2.1, we take L^∞ as the function space setting for the flexural rigidity coefficient. Furthermore, physical conditions require $a > 0$.

We further consider, for simplicity, only the boundary conditions

$$u(x) = h(x), \quad x \in \partial\Omega \tag{10}$$

$$M(x) = g(x), \quad x \in \partial\Omega \tag{11}$$

where

$$M(x) = a(x)\nabla^2 u(x), \quad x \in \overline{\Omega} \tag{12}$$

is the bending moment of the plate and g and h are given functions of x on $\partial\Omega$. When $g = h = 0$ on $\partial\Omega$ the boundary conditions (10) and (11) correspond to supported plates. More general, boundary conditions instead of (11), of the type

$$\alpha(x)M(x) + \frac{\partial M}{\partial n}(x) = g_1(x), \quad x \in \Gamma_1 \tag{13}$$

$$M(x) = g_2(x), \quad x \in \Gamma_2 \tag{14}$$

where $\alpha \geq 0$, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_2 \neq \emptyset$, can also be considered.

From eqns (3) and (12) we observe that M satisfies the Poisson equation in Ω, namely,

$$\nabla^2 M(x) = f(x), \quad x \in \Omega \tag{15}$$

which should be solved subject to the boundary condition (11).

In what follows, we assume that Ω is a connected, bounded domain with a sufficiently smooth boundary, e.g. satisfying the interior and/or exterior sphere condition, such as $\partial\Omega \in C^2$. From the theory of Poisson's equation we have the following theorem.

Theorem 2.1. *Let $f \geq 0$ be a bounded, locally Holder continuous function in Ω and $g \leq 0$ be a continuous function on $\partial\Omega$. Then the Poisson eqn.(15) subject to the Dirichlet boundary conditions (11) is uniquely solvable in $C^2(\Omega) \cap C(\overline{\Omega})$, and the following global estimate holds:*

$$\|M\|_\infty \leq \|g\|_{L^\infty(\partial\Omega)} + C\|f\|_\infty \tag{16}$$

where C is a positive constant depending only on the diameter of Ω. In particular, if Ω lies between two parallel planes which are a distance l apart then we can choose $C = e^l - 1$.
Furthermore, the following pointwise estimate holds:

$$0 \geq M(x) \geq Aw(x), \quad x \in \Omega \tag{17}$$

where

$$A = \max\left\{\|f\|_\infty, \|g\|_{L^\infty(\partial\Omega)}\right\} \tag{18}$$

and w is the unique solution of the problem

$$\begin{aligned} \nabla^2 w(x) &= 1, \quad x \in \Omega \\ w(x) &= -1, \quad x \in \partial\Omega. \end{aligned} \tag{19}$$

Moreover, if f and g are not simultaneously zero then $M < 0$ in Ω.
In the case where Ω is a ball $B = B_R(0)$ say, we have the explicit formula for the solution:

$$M(x) = -\int_{\partial B} K(x,y)g(y)dS_y - \int_B G(x,y)f(y)d\Omega_y, \quad x \in B \tag{20}$$

where K is the Poisson kernel given by

$$K(x,y) = \frac{R^2 - \mid x \mid^2}{d\omega_d R \mid x-y \mid^d}, \quad x \in B, \ y \in \partial B, \tag{21}$$

ω_d is the volume of the unit ball in R^d, and $G(x,y)$ is the Green function

$$G(x,y) = \begin{cases} -\Gamma(\mid x-y \mid) + \Gamma\left(\frac{|y|}{R} \mid x - \frac{R^2}{|y|^2}y \mid\right), & y \neq 0 \\ -\Gamma(\mid x \mid) + \Gamma(R), & y = 0 \end{cases} \tag{22}$$

where Γ is the normalized fundamental solution of Laplace's equation:

$$\Gamma(\mid x-y \mid) = \begin{cases} \frac{1}{d(2-d)\omega_d} \mid x-y \mid^{2-d}, & d > 2 \\ \frac{1}{2\pi} ln(\mid x-y \mid), & d = 2 \end{cases} \tag{23}$$

Proof. From [4, pp.35,55] it is well-known that the problem given by eqns (11) and (15) is uniquely solvable in $C^2(\Omega) \cap C(\overline{\Omega})$ and that the global estimate (16) and the representation formula (20) hold. We now prove the pointwise estimates (17). On using Green's identities the solution M has the representation

$$M(x) = -\int_\Omega G(y;x)f(y)d\Omega_y - \int_{\partial\Omega} g(y)\frac{\partial G}{\partial n_y}(y,x)dS_y, \quad x \in \Omega \tag{24}$$

where G is the Green's function for the Laplace equation with Dirichlet homogeneous boundary conditions on $\partial\Omega$. Using maximum principles it can be shown, [19, p.85], that the Green's function G has the properties

$$G > 0 \quad \text{in } \Omega, \quad \frac{\partial G}{\partial n} < 0 \quad \text{on } \partial\Omega. \tag{25}$$

Since $f \geq 0$ and $g \leq 0$ then it follows from eqn.(24) that $M \leq 0$. To obtain the lower bound for M from eqns (18) and (19), we remark that the function $z(x) = Aw(x)$ satisfies the inequalities

$$\begin{aligned} \nabla^2 z \geq f \quad & \text{in } \Omega \\ z \leq g \quad & \text{on } \partial\Omega. \end{aligned} \tag{26}$$

Defining $v = z - M$, we have that v satisfies the same inequalities as in eqns (26) and, on reasoning the same way as before, we find $v(x) \leq 0$ for $x \in \Omega$ and hence $M(x) \geq Aw(x)$ for all $x \in \Omega$.

Remark 2.2. (i) The existence of a unique solution of the Poisson problem with Dirichlet boundary conditions can be obtained in the weak sense $M \in H^1(\Omega)$ if $g \in H^{1/2}(\partial\Omega)$, $f \in H^{-1}(\Omega)$, or in the distribution sense $M \in H^{1/2}(\Omega)$ if $g \in L^2(\partial\Omega)$, $f \in H^{-3/2}(\Omega)$, [17, Chapter 2, Section 7.3].
(ii) The uniqueness of a classical negative solution for the mixed problem given by eqns (13)-(15) can be obtained using the same maximum principles, [19, p.85]. Further, this problem will always admit a weak or generalized solution, [3, p.209]; however even if the given functions f, g_1, g_2 are very regular, there will in general be discontinuities in the first derivatives of the solution M along the interface between Γ_1 and Γ_2, [23]. Ways to deal with the interface singularities can be found in [7,8].
(iii) Further well-posed results for the Robin problem given by eqns (13) and (15) when $\Gamma_1 = \partial\Omega$, $\Gamma_2 = \emptyset$ and $f \equiv 0$ can be found in [11].

3. CONTINUOUS DEPENDENCE OF THE FLEXURAL RIGIDITY

We now address the derivation of conditions under which eqn.(3) is guaranteed to have a unique solution a, and the characterisation of its dependence on the relevant parameters of the problem.

We first illustrate some essential features of the inverse problem by considering a one-dimensional example:

$$(a(x)u''(x))'' = f(x), \quad x \in \Omega \subset R^1. \tag{27}$$

This second-order ordinary differential equation can easily be integrated, yielding

$$a(x) = \frac{a(p)u''(p) + (x-p)(a(x)u''(x))'_{x=q} + F_1(x)}{u''(x)}, \quad x \in \Omega \tag{28}$$

where p, $q \in \overline{\Omega}$ and

$$F(x) = \int_q^x f(\xi)d\xi, \quad F_1(x) = \int_p^x F(\xi)d\xi. \tag{29}$$

If u'' is bounded away from zero over Ω, and the values of a and $(a(x)u''(x))'$ are given at some points p and $q \in \overline{\Omega}$, respectively, then it is obvious that a unique solution, as given by eqn.(28), exists for any integrable function f. It should be noted that if the shear force $(a(x)u''(x))'$ is prescribed at one point $q \in \partial\Omega$ as a boundary condition, then only a needs to be prescribed at a single point $p \in \overline{\Omega}$ for the existence and uniqueness of the solution. On the other hand, if u'' vanishes at a single point $p \in \overline{\Omega}$, and if u''' is bounded away from zero over Ω, then only the shear force $(a(x)u''(x))'$ needs to be specified at some point $q \in \overline{\Omega}$. In this case, the solution (28) at the point p is obtained by applying L'Hopital's rule as follows:

$$a(p) = \frac{(a(x)u''(x))'_{x=q} + F(p)}{u'''(p)}. \tag{30}$$

Further, if u'' and u''' vanish simultaneously at a single point $p \in \overline{\Omega}$ and u'''' is bounded away from zero over Ω, then a is well defined in Ω without any specification on a. In this degenerate case we have

$$a(p) = f(p)/u''''(p). \tag{31}$$

Finally, we observe that if u'', u''' and u'''' vanish simultaneously at a point $p \in \Omega$ (which, for the homogeneous version of the problem, will occur at any interior point where $u'' = u''' = 0$), we have a particularly ill-posed situation where $a(p)$ involves derivatives of the function u. Now suppose that u'' and u''' vanish simultaneously at several points $p_1 \leq p_2 \leq ...$, then a solution exists only if the compatibility conditions $\int_{p_i}^{p_{i+1}} f(\xi)d\xi = \int_{p_i}^{p_{i+1}} \left(\int_{p_i}^{x} f(\xi)d\xi\right) dx = 0$ are satisfied. In the case of the identification problem, where u arises as a response to f, the compatibility conditions are automatically satisfied, except for the presence of measurement errors.

In what follows, we let the input data (u, f) be given such that $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ and $f \in L^\infty(\Omega)$ is bounded and locally Holder continuous. As suggested by Theorem 2.1, in what follows we consider the inverse problem $L(a; u) = f$ for finding $a > 0$, $a \in L^\infty(\Omega)$, under the assumption that

$$sup_\Omega \nabla^2 u = -k < 0 \tag{32}$$

which circumvents the source of difficulty of the one-dimensional case.

Using Theorem 2.1 and Remark 2.2, the following identification result for the inverse problem is obtained.

Theorem 3.1. *Suppose that f and g are not simultaneously zero with $f \geq 0$ bounded and locally Holder continuous, $g \leq 0$ continuous and let eqn.(32) be satisfied. Then the problem $L(a; u) = f$ subject to (11) has a unique positive solution $a \in C^2(\Omega) \cap C(\overline{\Omega})$, which is explicitly given by*

$$a(x) = \frac{M(x)}{\nabla^2 u(x)}, \quad x \in \Omega. \tag{33}$$

Furthermore, a satisfies the pointwise estimate

$$a(x) \leq -\frac{A}{k} w(x), \quad x \in \Omega \tag{34}$$

and the global estimates

$$\|a\|_\infty \leq \frac{A}{k}\|w\|_\infty \leq \frac{A(1+C)}{k} \tag{35}$$

$$\|a\|_\infty \leq \frac{\|g\|_{L^\infty(\partial\Omega)} + C\|f\|_\infty}{k} \leq \frac{A(1+C)}{k}. \tag{36}$$

Proof. Firstly we note that since $f \geq 0$ and $g \leq 0$ are not simultaneously identically zero, from eqn.(24) and Theorem 2.1, then we have $M < 0$ as the unique solution in $C^2(\Omega) \cap C(\overline{\Omega})$ of the problem given by eqns (11) and (15). Further, since we have assumed that $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ and that the eqn.(32) is satisfied, we have that $a = M/\nabla^2 u \in C^2(\Omega) \cap C(\overline{\Omega})$ is well-defined and is strictly positive in Ω. Clearly, from eqn.(33), a is unique. From eqns (16), (17) and (33) we obtain the estimates (34)-(36), where in the last inequality in (35) we have applied (16) for the function w which satisfies the problem (19).

Example 3.1. Let us take $\Omega = \{(x, y) \mid x^2 + y^2 \leq 4\}$ to be the circle of radius 2, $f \equiv 1$, $g \equiv -1$, $u(x, y) = -(x^2 + y^2)/4$ and we aim to retrieve $a(x, y) = 2 - (x^2 + y^2)/4$. Solving the problem given by eqns (11) and (15) we obtain $M(x, y) = -2 + (x^2 + y^2)/4$ and solving the problem (19) we obtain $w(x, y) = -2 + (x^2 + y^2)/4$. Clearly, $\|f\|_\infty = 1$, $\|g\|_{L^\infty(\partial\Omega)} = 1$, $\|a\|_\infty = 2$, $A = 1$, $sup_\Omega \nabla^2 u = -k = -1$ and the estimate (35) gives $2 = \|a\|_\infty \leq \|w\|_\infty = 2$, whilst the estimate (36) gives $2 = \|a\|_\infty \leq 1 + C = e^2$, so this estimate is not so sharp.

At this stage, it is worth noting that the assumption (32) can be obtained directly by applying the maximum principle, in the form of replacing the condition $g \leq 0$ by $g < 0$, to the function M.

Next we establish the continuous dependence of a on the data f and g.

Theorem 3.2. *Let the assumptions of Theorem 3.1 be satisfied, and let a and $\overline{a}$ be solutions of $L(a; u) = f$ in Ω, $M = g$ on $\partial\Omega$ and $L(\overline{a}; u) = \overline{f}$ in Ω, $\overline{M} = \overline{g}$ on $\partial\Omega$, respectively. Then the retrieval of a is stable with respect to noise in the data f and g, and we have the estimates*

$$| a(x) - \overline{a}(x) | \leq -\frac{\epsilon}{k} w(x), \quad x \in \Omega \tag{37}$$

$$\|a - \overline{a}\|_\infty \leq \frac{\epsilon}{k}\|w\|_\infty \leq \frac{\epsilon(1 + C)}{k} \tag{38}$$

where $a = \frac{M}{\nabla^2 u}$, $\overline{a} = \frac{\overline{M}}{\nabla^2 u}$, M and $\overline{M}$ are the solutions of the problem (11) and (15) with the data f, g and $\overline{f}$, $\overline{g}$, respectively, and

$$\epsilon = max\{\|f - \overline{f}\|_\infty, \|g - \overline{g}\|_{L^\infty(\partial\Omega)}\} \tag{39}$$

Proof. The inequality (17) gives $| M(x) | \leq -Aw(x)$ for all $x \in \Omega$. Then repeating the same arguments based on maximum principles we obtain

$$| M(x) - \overline{M}(x) | \leq -\epsilon w(x), \quad x \in \Omega \tag{40}$$

and hence the error estimates (37) and (38) follow immediately from the fact that $a = \frac{M}{\nabla^2 u}$ and $\overline{a} = \frac{\overline{M}}{\nabla^2 u}$.

Now suppose that $L(b; v) = \tilde{f}$, where b is the flexural rigidity coefficient produced by a perturbed forward solution $v \approx u$ and loading function $\tilde{f} \approx f$. On denoting $z := L(b; v - u) + f - \tilde{f}$ and applying Theorem 3.1 to the problem $L(a - b; u) = z$, we obtain the following stability result.

Theorem 3.3. *Let the assumption (32) be satisfied. If $L(a; u) = f$ and $L(b; v) = \tilde{f}$, $z > 0$ in Ω and $(a - b)\nabla^2 u \mid_{\partial\Omega} = 0$ then*

$$\|a - b\|_\infty \leq \frac{\|w\|_\infty}{k} B \leq \frac{(1 + C)B}{k} \tag{41}$$

where

$$B = \|f - \tilde{f}\|_\infty + \|\nabla^2 b\|_\infty \|\nabla^2(v - u)\|_\infty + 2\|\nabla b\|_\infty \|\nabla^3(v - u)\|_\infty + \|b\|_\infty \|\nabla^4(v - u)\|_\infty \tag{42}$$

Remark 3.1. Theorem 3.3 shows that the retrieval of a is stable with respect to noise in the data u and f. However, it suggests that a successful identification of a will be feasible only if the observed u is sufficiently precise to permit an accurate approximation of the fourth derivative of u. However, it should be noted from eqn.(28) that a varies as the second derivative of u in the one-dimensional case.
To see that this situation is anomalous, consider the following example in the two-dimensional unit circle, namely:

Exmaple 3.2. Consider

$$a = x^2 + y^2, \quad u = -(x^2 + y^2)/16, \quad f = -1 \tag{43}$$

$$b = 1, \quad v = -(x^4 + y^4)/48, \quad \tilde{f} = -1 \tag{44}$$

Here $\|a - b\|_\infty = 1$, $\|\nabla^2(v - u)\|_\infty = 1/4$, $\|\nabla^3(v - u)\|_\infty = 1/2$, $\|\nabla^4(v - u)\|_\infty = 1$, $k = 1/4$, $M = -(x^2 + y^2)/4$. Hence from eqn.(42) we obtain $B = \|\nabla^4(v - u)\|_\infty = 1$. The bound (41) is readily seen to be sharp for this example, and it is clear that $\|a - b\|_\infty$ is not bounded in terms of $\|\nabla^2(v - u)\|_\infty$.

The inequality (41) can be used to obtain an *a posteriori* estimate of the accuracy of b provided the errors in $\nabla^2 v$, $\nabla^3 v$, $\nabla^4 v$ and $\tilde{f}$ are available. In any case, it characterises the dependence of $\|a - b\|_\infty$ on these errors. In practice it is usually most appropriate to view the approximate flexural rigidity coefficient b as an intermediate quantity, and judge its accuracy in terms of the results it yields for the forward problem under conditions which are different from those which prevailed during identification. Accordingly, we consider an approximation $b \approx a$, as in Theorem 3.3, and a "subsequent" forward problem together with its perturbed counterpart, namely, $L(a, \phi) = L(b, \overline{\phi})$ subject to the boundary conditions (10) and (11) corresponding to supported plates. The error in $\overline{\phi}$ is thus solely due to that in b. Assuming $b > 0$ and combining Lemma 2.1 and Theorem 3.3 we obtain the following corollary.

Corollary 3.1. *Let the assumption (32) be satisfied. If* $L(a; \phi) = L(b; \overline{\phi})$ *satisfy the same boundary conditions (10) and (11) with the same* h *and* g *and* $z := L(b; \phi - \overline{\phi}) > 0$ *in* Ω *and* $(a - b)\nabla^2\phi \mid_{\partial\Omega} = 0$, *then*

$$\frac{\|\nabla^2(\phi - \overline{\phi})\|_2}{\|\nabla^2\phi\|_2} \leq \frac{(C+1)B}{k\ inf_\Omega b}. \tag{45}$$

4. IDENTIFIABILITY OF A HARMONIC FLEXURAL RIGIDITY

In this section we consider a particular case in which the flexural rigidity coefficient a is a harmonic function, i.e. $\nabla^2 a = 0$. This class includes the important cases of constant and linear spacewise dependent flexural rigidities of plates. In such a situation, eqn.(3) can be written as a hyperbolic equation for a, namely

$$L_1(a; u) := 2\nabla a \bullet \nabla^3 u + a\nabla^4 u = f \tag{46}$$

and the mathematical and numerical analyses of [20,21] for the steady-state diffusion equation $\nabla a \bullet \nabla u + a\nabla^2 u = f$ can be applied. Here we mention only the main results.

Theorem 4.1. *Suppose that* $\nabla^2 a = 0$ *in* Ω *and that* Ω *can be divided into two subregions* Ω_1 *and* Ω_2 *such that*

$$\mid \nabla^3 u \mid \geq k_1 > 0 \quad \text{in } \Omega_1, \quad \nabla^4 u \geq k_2 > 0 \quad \text{in } \Omega_2. \tag{47}$$

Then for any $f \in L^\infty(\Omega)$, *the problem* $L_1(a; u) = f$ *has at most one solution* $a \in \Lambda_{ad} = L^\infty(\Omega) \cap C^1(\Omega_2)$ *assuming prescribed values along the "inflow" portion* Γ *of the boundary* $\partial\Omega$ *(essentially that portion* $\Gamma \subset \partial\Omega$ *where the outward normal derivative of* $\nabla^2 u$ *is negative), and*

$$\|a\|_\infty \leq C_1 \left[max\left\{ \|a\|_{L^\infty(\partial\Omega)}, \frac{\|f\|_\infty}{k_2} \right\} + \frac{[\nabla^2 u]\|f\|_\infty}{k_1^2} \right] \tag{48}$$

where

$$[\nabla^2 u] = sup_\Omega \nabla^2 u - inf_\Omega \nabla^2 u, \quad q_1 = sup_{\Omega_1}\left\{ -\frac{\nabla^4 u}{2 \mid \nabla^3 u \mid} \right\},$$

$$C_1 = max\left\{ 1, exp\left(\frac{q_1[\nabla^2 u]}{k_1} \right) \right\}. \tag{49}$$

Next we give a particularly advantageous set of test conditions for observing the forward solution. For a given load function f and boundary conditions on $\partial\Omega$, a solution u to the elliptic forward problem (3) is observed. This (f, u)-pair is then used in solving the problem $L_1(a; u) = f$ for the unknown flexural rigidity coefficient. We shall be concerned with the following "test conditions" for the forward problem:
(i) f is positive and Holder continuous in $\overline{\Omega}$.
(ii) $u = h = M = g = 0$ on $\partial\Omega$ (supported plates). The condition $M = 0$ on $\partial\Omega$ also violates the previously imposed condition (32).
(iii) $a \in C^2(\Omega) \cap C(\overline{\Omega})$ is positive and $\nabla^2 a = 0$.
(iv) Ω lies between two parallel planes a distance l apart and satisfies the "exterior sphere condition",

i.e. for any $P \in \partial\Omega$, there exists a ball B_Q centered at $Q \notin \overline{\Omega}$ such that $\overline{B_Q} \cap \overline{\Omega} = P$.

Theorem 4.2. *If u arises following the test conditions (i)-(iv), the problem $L_1(b;u) = \tilde{f}$ has at most one harmonic solution $b \in L^\infty(\Omega)$ for any $\tilde{f} \in L^\infty(\Omega)$. Furthermore,*

$$\|b\|_\infty \leq H(a,\Omega,f)\|\tilde{f}\|_\infty \tag{50}$$

where

$$H(a,\Omega,f) = \frac{a_{min}}{f_{min}}\, inf_{\theta\in(0,1)}\left\{\left(\frac{D}{1-\theta} + \frac{E}{\theta^2}\right) exp\left(\frac{E}{\theta}\right)\right\} \tag{51}$$

$$0 < a_{min} = min_{\overline{\Omega}} a, \quad D = \frac{\|a\|_\infty}{a_{min}}, \quad 0 < f_{min} = min_{\overline{\Omega}} f,$$
$$E = \frac{\|\nabla a\|^2_\infty \|f\|_\infty (e^l - 1)}{a^2_{min} f_{min}} \tag{52}$$

It can also be shown that the stability bound (20) is potentially sharp in the case $a = constant$. In this case $E = 0$, so $H(a,\Omega,f) = a/f_{min}$ and (50) becomes $\|b\|_\infty/a \leq \|\tilde{f}\|/f_{min}$. Note that in the case when the flexural rigidity is harmonic and constant then $L(a;u) = f$ reduces to $a\nabla^4 u = f$, and also the harmonic function b satisfies $2\nabla b \cdot \nabla^3 u + b\nabla^4 u = \tilde{f}$. If we choose b so that ∇b is orthogonal to ∇^3, then we obtain $b\nabla^4 u = \tilde{f}$. From this we infer that $b(P)/a = \tilde{f}(P)/f(P)$. Thus $\|b\|_\infty/a \leq \|\tilde{f}\|/f_{min}$ will be sharp if $\tilde{f}$ is largest at the same point where f is smallest.

Finally, the stability bound (50) is potentially useful in assessing the inaccuracy in a resulting from measurement error in f and u, as we now indicate. Indeed, applying theorem 4.2 to the identity $L((\overline{a} - a); u) = -L(\overline{a}; \overline{u} - u) + (\overline{f} - f)$ we obtain the following corollary.

Corollary 4.1 *If $\overline{a}$ arises as the solution of the perturbed problem $L_1(\overline{a};\overline{u}) = \overline{f}$ where $\overline{u} \approx u$ and $\overline{f} \approx f$, then under the test conditions (i)-(iv) we have*

$$\|\overline{a} - a\|_\infty \leq H(a,\Omega,f)\epsilon \tag{53}$$

where

$$\epsilon = \|f - \overline{f}\|_\infty + 2\|\nabla\overline{a}\|_\infty \|\nabla^3(\overline{u} - u)\|_\infty + \|\overline{a}\|_\infty \|\nabla^4(\overline{u} - u)\|_\infty \tag{54}$$

This can be used to obtain an a posteriori estimate of the accuracy of $\overline{a}$ provided the errors in $\nabla^3\overline{u}$, $\nabla^4\overline{u}$ and $\overline{f}$ are quantifiable.

Finally, if we consider the direct problems $L(a;w) = L(\overline{a}, \overline{w})$ subject to the boundary conditions given in Lemma 2.1 and using corollary 4.1 we obtain the continuous dependence of the direct problem as given by the estimate

$$\frac{\|\nabla(\overline{w} - w)\|_2}{\|\nabla w\|_2} \leq \frac{H(a,\Omega,f)\epsilon}{inf_\Omega \overline{a}} \tag{55}$$

provided that $\overline{a} > 0$.

5. CONCLUSIONS AND FUTURE WORK

In this paper some mathematical aspects of the inverse problem of identifying the spatially varying flexural rigidity of a plate using an observed deflection solution of a dynamic linear Kirchhoff plate model have been addressed. In particular, it has been shown that the flexural rigidity coefficient can be identified if a non-negative load and natural boundary conditions expressing negative functions are prescribed on the plate. In addition, if the deflection can be measured such that its Laplacian is negative then stability estimates for the flexural rigidity coefficient have been established. This assumption can be removed in the situation when the flexural rigidity is a harmonic function, in which case the uniqueness and stability of an associated problem have been established.

Nevertheless, in the infinite-dimensional setting of this study the uniqueness of the coefficient present in the partial differential eqn.(3) followed from regularity properties of involved functions, similarly as obtained in [5] in a result concerning identifiability of the inverse problem of groundwater hydrology.

However, these properties disappear in the discrete case, [18]. Therefore, in order to take into account the case of pointwise measurements of u, further work will be concerned with the numerical identification of the spatially varying flexural rigidity coefficient. This task can be accomplished by recasting eqns (12) and (15) as an ill-posed Fredholm integral equation of the first kind and employing Green's formula combined with Tikhonov's regularization method for minimizing with repect to $\alpha = a^{-1}$ the linear functional

$$\int_\Omega \left\{ \int_\Omega T(x,y)M(y)\alpha(y)d\Omega_y + u(x) - \int_{\partial\Omega} \left[T(x,y)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial T}{\partial n_y}(x,y) \right] dS_y \right\}^2 d\Omega_x + \lambda \int_\Omega \alpha^2(x) d\Omega_x \qquad (56)$$

subject to the constraint $\alpha > 0$, where $\lambda \geq 0$ is a regularization parameter, T is the fundamental solution for the Laplace equation and $M(x)$ satisfies the integral equation

$$\eta(x)M(x) = \int_{\partial\Omega} \left[T(x,y)\frac{\partial M}{\partial n}(y) - M(y)\frac{\partial T}{\partial n_y}(x,y) \right] dS_y - \int_\Omega f(y)T(x,y)d\Omega_y, \quad x \in \overline{\Omega} \qquad (57)$$

where $\eta(x)$ is a coefficient function which is equal to 1 if $x \in \Omega$ and 0.5 if $x \in \partial\Omega$ (smooth).

This approach resembles an extension to two-dimensions of the one-dimensional work on beams, [2]. Alternatively, if the flexural rigidity is harmonic, as in section 4, one can extend the numerical method of [21] related to hyperbolic problems.

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REFERENCES

1. H.T. Banks, S.S. Gates, I.G. Rosen and Y. Wang, The identification of a distributed parameter model for a flexible structure. *SIAM J. Control Optimiz.* (1988) **26**, 743-762.

2. W.D. Collins, S. Quegan, D. Smith and D.A.W. Taylor, Using Bernstein polynomials to recover engineering loads from strain gauge data. *Q. Jl Mech. appl. Math.* (1994) **47**, 527-539.

3. R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol.**1**, Springer-Verlag, Berlin, 1990.

4. D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1977.

5. M. Giudici, A result concerning identifiability of the inverse problem of groundwater hydrology. *Inverse Problems* (1989) **5**, L31-L36.

6. M. Grediac and P.A. Paris, Direct identification of elastic constants of anisotropic plates by modal analysis: Theoretical and numerical aspects. *J. Sound Vibration* (1996) **195**, 401-415.

7. P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, 1985.

8. P. Grisvard, *Singularities in Boundary Value Problems*, Masson, Paris, 1992.

9. A. Hasanov and Z. Seyidmamedov, Determination of unknown elastoplastic properties in Signorini problem. *Int. J. Non-Linear Mechanics* (1998) **33**, 979-991.

10. M. Ikehata, An inverse problem for the plate in Love-Kirchhoff theory. *SIAM J. Appl. Math.* (1993) **53**, 942-970.

11. G. Inglese, An inverse problem in corrosion detection, *Inverse Problems* (1997) **13**, 977-994.

12. I. Ismailov and L. Muravey, Some problems of coefficient control for elliptic equations of high order, In *Inverse and Ill-Posed Problems (Abstracts)*, Moscow, 1996, 81.

13. K. Ito and S. Nakagiri, Identifiability of stiffness and damping coefficients in Euler-Bernoulli beam equations with Kelvin-Voigt damping. *Numer. Funct. Anal. Optimiz.* (1997) **18**, 107-129.

14. S.G. Lekhnitskii, *Anisotropic Plates*, Gordon & Breach, New York, 1968.

15. D. Lesnic, L. Elliott and D.B. Ingham, Analysis of coefficient identification problems associated to the inverse Euler-Bernoulli beam theory. *IMA J. Appl. Math.* (1999) **62**, 101-116.

16. D. Lesnic, Identifiability of distributed parameters in beam-type systems. *J. Inverse Ill-Posed Problems* (2000) **8**, 379-397.

17. J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, Vol.**1**, Springer-Verlag, Berlin, 1972.

18. G. Ponzini and G. Crosta, The comparison model method: a new arithmetic approach to the discrete inverse problem of groundwater hydrology. 1. One-dimensional flow. *Transport in Porous Media* (1988) **3**, 415-436.

19. M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1967.

20. G.R. Richter, An inverse problem for the steady state diffusion equation. *SIAM J. Appl. Math.* (1981) **41**, 210-221.

21. G.R. Richter, Numerical identification of a spatially varying diffusion coefficient. *Math. Comput.* (1981) **36**, 375-386.

22. S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill, New York, 1959.

23. W.L. Wendland, E. Stephan and G.C. Hsiao, On the integral method for the plane mixed boundary value problem of the Laplacian. *Math. Meth. Appl. Sci.* (1979) **1**, 265-321.

24. L.W. White, Identification of flexural rigidity in a dynamic plate model, *J. Math. Anal. Appl.* (1989) **144**, 275-303.

25. H.S. Zibdeh and R. Rackwitz, Moving loads on beams with general boundary conditions. *J. Sound Vibr.* (1996) **195**, 85-102.